

Entanglement induced Sub-Planck structures

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We study Wigner function of a system describing entanglement of two superposed coherent-states. Quantum interference arising due to entanglement is shown to produce sub-Planck structures in the phase-space plots of the Wigner function. Origin of these structures in our case depends on entanglement unlike those in Zurek [1]. It is argued that these kind of entangled compass states are better suited for carrying out precision measurements.

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Recently it has been demonstrated that, the quantum states obtained from superposition of coherent states can be useful in quantum metrology, especially in carrying out Heisenberg-limited measurements and quantum parameter estimation [1,2]. Zurek has shown that non-local superposition of coherent quantum states can have well-defined oscillatory structures in the phase space, at scales smaller than Planck constant \hbar . More surprisingly, contrary to the commonly held belief, these structures can be physically important. The Wigner distribution function for a compass state, $|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle$ has checker-board type of structure in the phase space due to quantum interference. Here α is a complex parameter used for characterizing the coherent states and its magnitude, in the present context, signifies a distance from the origin in the phase-space. The typical area a of the fundamental tile of the checker-board is $a = \hbar \frac{\hbar}{\mathcal{A}}$ where, \mathcal{A} is the area of the accessible phase space, which can be estimated from the total energy. The sub-Planck structures arise because for a compass state with superposition of well-separated coherent states in the phase-space, $\frac{\hbar}{\mathcal{A}} \ll 1$ and thus $a \ll \hbar$. Locations of the coherent states in the phase-space of the compass state can be denoted by, intuitively obvious, geographical notations namely north (N), south (S), east (E) and west (W), to denote their relative positions in the phase space. It can be shown that the interference between $NS(|i\alpha\rangle + |-i\alpha\rangle)$ and $EW(|-\alpha\rangle + |\alpha\rangle)$ combinations produce the checker-board type of pattern [1]. If an act of measurement or any other process displaces the original compass state by a distance \sqrt{a} in the phase space, the sub-Planck structures allow the resultant state to be distinguished from the old one. i.e., both the states become approximately orthogonal. Thus such states are argued to be sensitive to external perturbations. Quite naturally they can have implications for processes like decoherence [1]. These states are useful in carrying out Heisenberg-limited sensitive measurements [2]. It ought to be noted here that a simpler state involving just two superposed coherent states can also be used for carrying out these measurements. However such states offer sensitivity to the perturbations in only one direction i.e., in the direction perpendicular to the line joining the two coherent states [2]. The Wigner function for these kind of states do not show the checker-board pattern in phase space. In comparison the compass state, can offer sensitivity in all directions of the phase-space.

The sub-Planck structures have been studied by various researchers to further investigate their properties and test some of the assumptions made. The issue of sensitivity to external perturbation of these structures, as the system evolves with time, was studied using Loschmidt echoes [3]. However the results of this work remain inconclusive. Recently it has been demonstrated that the above compass states, due to their Heisenberg-limited sensitivity to external perturbation can be utilized for quantum parameter estimation [2]. In this work, the state was entangled to a two-level atomic system for carrying out the measurements. Generation of the compass state in cavity QED and its decoherence characteristics were studied in Ref.[4]. Such states can also be generated during the fractional revival process of molecular wave-packets [5]. A classical analog of these states was found in Ref.[6]. It was shown that two pulses displaced by a small sub-Fourier shift of the carrier frequency become mutually orthogonal. The single particle compass state used in Ref.[1] is composed of four coherently superposed localized Gaussians. These states are rather difficult to produce. It has been shown in Ref.[4] that, the interference pattern (sub-Planckian structures) arising due to the superposition of all the four Gaussians (NS and EW combinations) can disappear faster than the interference pattern between any two Gaussians due to decoherence. In this work, we aim to study the sub-Planck structures in the phase space of a bipartite system and analyse the role of entanglement in it. For this goal, we propose a new

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compass state $|\psi\rangle_c$ consisting of the following:

$$|\psi\rangle_c = \frac{1}{\sqrt{2}}(A|\pm\alpha\rangle_1|\pm\alpha\rangle_2 + B|\pm\alpha\rangle_1|\pm\alpha\rangle_2), \quad (1)$$

where, $A = A_1 + iA_2$ and $B = B_1 + iB_2$ are complex parameters that control the entanglement. The states in Eq.(1) are given by,

$$|\pm\alpha\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle). \quad (2)$$

The choice of the compass state $|\psi\rangle_c$ is such that, when one considers the Wigner function for a constituent particle state (i.e., for state like in Eq.(1)), it does not show any checker-board pattern. But the Wigner function for the entire $|\psi\rangle_c$ shows these structures. It can be demonstrated that they arise solely due to entanglement. Since the degree of entanglement in the state given by Eq.(1) is determined by A and B only and does not depend on α , there are two different decoherence characteristics for the superposition and the entanglement. Superposition of the coherent state is strongly affected by noise process say absorption of photons, while it can have no effect on the entanglement [7]. The entangled coherent states are more robust against the decoherence arising due to photon absorptions noise. Thus keeping in mind the results of Refs.[4, 7], the state proposed by us can be more suitable for carrying out Heisenberg-limited measurements. Since the proposed entangled states give sub-Planck structures in the Wigner function, they offer sensitivity in all the directions in the phase-space.

We represent the states states (given in Eq.(1)) by localized coherent Gaussian states, to construct the normalized coordinate $|\pm\alpha\rangle \rightarrow \psi(x)$ and the momentum even states $|\pm i\alpha\rangle \rightarrow \varphi(x)$:

$$\psi(x) = \frac{e^{-(x+x_0)^2/2\delta^2} + e^{-(x-x_0)^2/2\delta^2}}{\sqrt{2\pi}^{1/4}\delta^{1/2} [1 + e^{-x_0^2/\delta^2}]^{1/2}} \quad (3)$$

and,

$$\varphi(x) = \frac{e^{-x^2/2\delta^2 + ip_0x/\hbar} + e^{-x^2/2\delta^2 - ip_0x/\hbar}}{\sqrt{2\pi}^{1/4}\delta^{1/2} [1 + e^{-p_0^2\delta^2/\hbar^2}]^{1/2}}, \quad (4)$$

where, x_0 , p_0 and δ are taken to be real quantities. Superposition of the states given by Eqs.(3-4) can give the representation of the single particle compass state considered in Zurek [1]. The compass state proposed here is given by,

$$\Psi(x_1, x_2) = \mathcal{N}[A\psi(x_1)\phi(x_2) + B\phi(x_1)\psi(x_2)] \quad (5)$$

where, \mathcal{N} is the normalization constant. It should be mentioned that non-separability condition for the wavefunctions of continuous variables is not fully established. The state in Eq.(5) does not satisfy separability criterion based on the variance approach [8, 9].

From Eq.(5) the correlation function is obtained,

$$c(x_1, a_1, x_2, a_2) = \Psi^\dagger\left(x_1 + \frac{a}{2}, x_2 + \frac{b}{2}\right) \Psi\left(x_1 - \frac{a}{2}, x_2 - \frac{b}{2}\right). \quad (6)$$

The Wigner function, in four dimensional phase-space, can then be defined as,

$$W(x_1, p_1; x_2, p_2) = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(x_1, a_1, x_2, a_2) e^{\frac{i(p_1 a_1 + p_2 a_2)}{\hbar}} da_1 da_2. \quad (7)$$

A lengthy calculation yeilds

$$W(x_1, p_1; x_2, p_2) = \frac{2\delta^2 c |\mathcal{N}|^2}{\pi \hbar^2} e^{-\frac{(x_1^2 + x_2^2)}{\delta^2} - \frac{(p_1^2 + p_2^2)\delta^2}{\hbar^2}} (W_{D1} + W_{D2} + e^{-\frac{x_0^2}{2\delta^2} - \frac{p_0^2\delta^2}{2\hbar^2}} (W_{C1} + W_{C2})), \quad (8)$$

where, W_{D1} , W_{D2} and W_{C1} , W_{C2} are, respectively, the diagonal and off-diagonal components of the Wigner function.

First consider one of the diagonal terms,

$$\begin{aligned}
W_{D1} = & 2|A|^2 \left(e^{-\frac{x_0^2}{\delta^2} - \frac{p_0^2 \delta^2}{\hbar^2}} \cosh\left(\frac{2p_0 p_2 \delta^2}{\hbar^2}\right) \cosh\left(\frac{2x_0 x_1}{\delta^2}\right) + e^{-\frac{x_0^2}{\delta^2}} \cosh\left(\frac{2x_0 x_1}{\delta^2}\right) \cos\left(\frac{2p_0 x_2}{\hbar}\right) + \right. \\
& \left. e^{-\frac{p_0^2 \delta^2}{\hbar^2}} \cos\left(\frac{2x_0 p_1}{\hbar}\right) \cosh\left(\frac{2p_0 p_2 \delta^2}{\hbar^2}\right) + 2 \cos\left(\frac{2p_0 x_2}{\hbar}\right) \cos\left(\frac{2x_0 p_1}{\hbar}\right) \right). \quad (9)
\end{aligned}$$

It can be seen from above that the first three terms containing hyperbolic functions are multiplied by constant Gaussian factors which are bound to be small, in the present mesoscopic context concerned with relatively larger values of x_0 and p_0 . Thus only the last term in Eq.(9) becomes dominant in the region between the Gaussians. This term is a purely oscillating term, which can produce significant amount of interference. The zeroes of this term occur at $x_2 = \pm \frac{\pi \hbar}{4p_0}$ and $p_1 = \pm \frac{\pi \hbar}{4x_0}$ from which one can calculate the fundamental area of the tile as $\frac{(2\pi \hbar)^2}{4x_0 p_0}$. It should be noted that Eq.(9) has a $|A|^2$ factor and the above calculation does not require any information about entanglement. Indeed one can see sub-Planck structure in this plane even when $|B| = 0$. As this plane has mixed coordinates i.e., momentum p_1 of particle one and position x_2 of particle two, we believe that the structures observed here are not physically important.

Now we compute the other diagonal term which has $|B|^2$ as a factor,

$$\begin{aligned}
W_{D2} = & 2|B|^2 \left(e^{-\frac{x_0^2}{\delta^2} - \frac{p_0^2 \delta^2}{\hbar^2}} \cosh\left(\frac{2p_0 p_1 \delta^2}{\hbar^2}\right) \cosh\left(\frac{2x_0 x_2}{\delta^2}\right) + e^{-\frac{x_0^2}{\delta^2}} \cosh\left(\frac{2x_0 x_2}{\delta^2}\right) \cos\left(\frac{2p_0 x_1}{\hbar}\right) + \right. \\
& \left. e^{-\frac{p_0^2 \delta^2}{\hbar^2}} \cos\left(\frac{2x_0 p_2}{\hbar}\right) \cosh\left(\frac{2p_0 p_1 \delta^2}{\hbar^2}\right) + 2 \cos\left(\frac{2p_0 x_1}{\hbar}\right) \cos\left(\frac{2x_0 p_2}{\hbar}\right) \right). \quad (10)
\end{aligned}$$

This has a similar structure as that of W_{D1} except that it has two Gaussians located at $x_2 = x_0$ and $p_1 = p_0$. From the argument given above one can see the sub-Planck structures in the $x_1 p_2$ -plane. Once again they may not be physically relevant.

Next we examine the off-diagonal terms, which can be computed to be

$$\begin{aligned}
W_{C1} = & ((A_1 B_1 + A_2 B_2) - (A_1 B_2 - A_2 B_1)) \left(e^{\frac{ip_0 x_0}{\hbar}} \left(\cosh\left(\left(\frac{x_0}{\delta^2} - \frac{ip_0}{\hbar}\right)(x_1 + x_2) + \left(\frac{ix_0}{\hbar} - \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 - p_2)\right) \right) + \right. \\
& \left. \cosh\left(\left(\frac{x_0}{\delta^2} - \frac{ip_0}{\hbar}\right)(x_1 - x_2) + \left(\frac{ix_0}{\hbar} - \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 + p_2)\right) \right) + \\
& e^{-\frac{ip_0 x_0}{\hbar}} \left(\cosh\left(\left(\frac{x_0}{\delta^2} + \frac{ip_0}{\hbar}\right)(x_1 + x_2) + \left(\frac{ix_0}{\hbar} + \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 - p_2)\right) \right) + \\
& \left. \cosh\left(\left(\frac{x_0}{\delta^2} + \frac{ip_0}{\hbar}\right)(x_1 - x_2) + \left(\frac{ix_0}{\hbar} + \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 + p_2)\right) \right) + \\
& 2 \left(\cos\left(p_0 \left(\frac{(x_1 - x_2)}{\hbar} - \frac{i(p_1 + p_2) \delta^2}{\hbar^2}\right)\right) \cosh\left(x_0 \left(\frac{(x_1 + x_2)}{\delta^2} + \frac{i(p_1 - p_2)}{\hbar}\right)\right) + \right. \\
& \left. \cos\left(p_0 \left(\frac{(x_1 + x_2)}{\hbar} - \frac{i(p_1 - p_2) \delta^2}{\hbar^2}\right)\right) \cosh\left(x_0 \left(\frac{(x_1 - x_2)}{\delta^2} + \frac{i(p_1 + p_2)}{\hbar}\right)\right) \right) \quad (11)
\end{aligned}$$

and,

$$\begin{aligned}
W_{C2} = & ((A_1 B_1 + A_2 B_2) + (A_1 B_2 - A_2 B_1)) \left(e^{\frac{ip_0 x_0}{\hbar}} \left(\cosh\left(\left(\frac{x_0}{\delta^2} - \frac{ip_0}{\hbar}\right)(x_1 + x_2) - \left(\frac{ix_0}{\hbar} - \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 - p_2)\right) \right) + \right. \\
& \left. \cosh\left(\left(\frac{x_0}{\delta^2} - \frac{ip_0}{\hbar}\right)(x_1 - x_2) - \left(\frac{ix_0}{\hbar} - \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 + p_2)\right) \right) + \\
& e^{-\frac{ip_0 x_0}{\hbar}} \left(\cosh\left(\left(\frac{x_0}{\delta^2} + \frac{ip_0}{\hbar}\right)(x_1 + x_2) - \left(\frac{ix_0}{\hbar} + \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 - p_2)\right) \right) + \\
& \left. \cosh\left(\left(\frac{x_0}{\delta^2} + \frac{ip_0}{\hbar}\right)(x_1 - x_2) - \left(\frac{ix_0}{\hbar} + \frac{p_0 \delta^2}{\hbar^2}\right)(p_1 + p_2)\right) \right) +
\end{aligned}$$

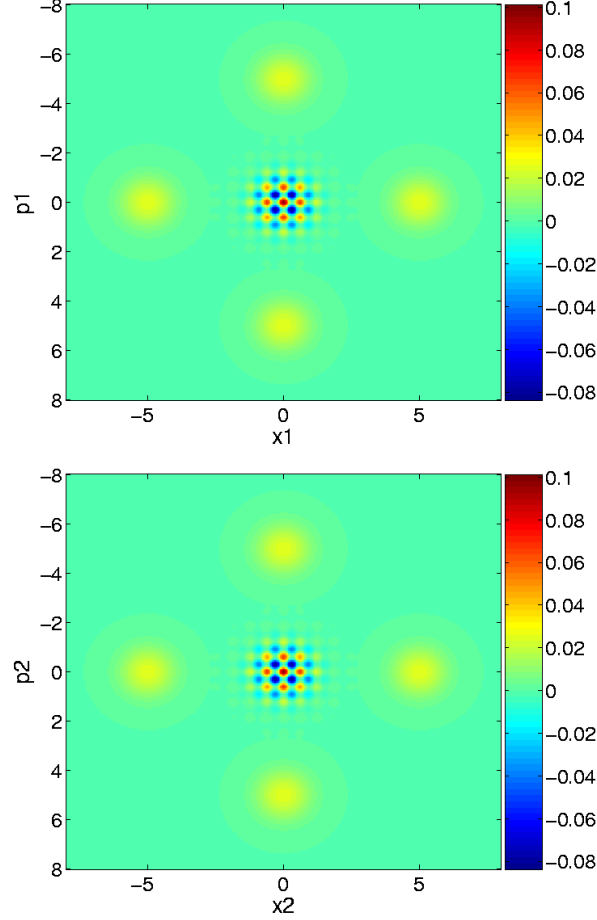


FIG. 1: (color online) Cross-sectional view of Wigner function of entangled wavefunction with $A_1 = A_2 = \frac{1}{\sqrt{2}}$ and $B_1 = -B_2 = \frac{1}{\sqrt{2}}$ in (a) $(x_1 p_1)$ plane, with $x_2 = 0$, $p_2 = 0$ and (b) $(x_2 p_2)$ plane with $x_1 = 0$, $p_1 = 0$.

$$2\left(\cos\left(p_0\left(\frac{(x_1 - x_2)}{\hbar} + \frac{i(p_1 + p_2)\delta^2}{\hbar^2}\right)\right) \cosh\left(x_0\left(\frac{(x_1 + x_2)}{\delta^2} - \frac{i(p_1 - p_2)}{\hbar}\right)\right) + \cos\left(p_0\left(\frac{(x_1 + x_2)}{\hbar} + \frac{i(p_1 - p_2)\delta^2}{\hbar^2}\right)\right) \cosh\left(x_0\left(\frac{(x_1 - x_2)}{\delta^2} - \frac{i(p_1 + p_2)}{\hbar}\right)\right)\right). \quad (12)$$

Interestingly one sees the presence of EPR variables in each term. However, one finds that although purely oscillatory terms are present here, they are significantly damped as compared to the diagonal terms for large values of x_0 and p_0 . This is clearly evident from the Wigner function.

From the discussion so far, one may wonder if it is possible at all to see checker-board type sub-Planck structures in $x_1 p_1$ or $x_2 p_2$ planes. Answer to this question can be found by adding the oscillatory terms from Eqs.(9-10),

$$4|A|^2 \cos\left(\frac{2p_0 x_2}{\hbar}\right) \cos\left(\frac{2x_0 p_1}{\hbar}\right) + 4|B|^2 \cos\left(\frac{2p_0 x_1}{\hbar}\right) \cos\left(\frac{2x_0 p_2}{\hbar}\right). \quad (13)$$

From the above, distance between two zeros in x_1 direction is again $\pm \frac{\pi\hbar}{4p_0}$ while it is $\pm \frac{\pi\hbar}{4x_0}$ in p_1 -direction. This gives the area of the fundamental tile $a = \frac{(2\pi\hbar)^2}{4x_0 p_0}$ in $x_1 p_1$ plane of particle one. Similarly one can find zeros in x_2 and p_2 directions and obtain the same value of the fundamental area. It should be noted that fundamental area a , though does not depend upon A or B , both of them need to be simultaneously non zero in order to get sub-Planck structures in the physical $x_1 p_1$ or $x_2 p_2$ plane. It is clear that visibility of the interference patterns depends upon the relative magnitudes of A and B . In Figs.(1-2) we have shown plots of the Wigner function (Eq.(8)) in $x_1 p_1$ and $x_2 p_2$ planes. Fig.(1) depicts cross-sectional view of the Wigner function in $x_1 p_1$ and $x_2 p_2$ planes while keeping A and B both non zero. It clearly shows the checker-board type pattern with $a \ll \hbar$. Area of the fundamental tile matches with a that

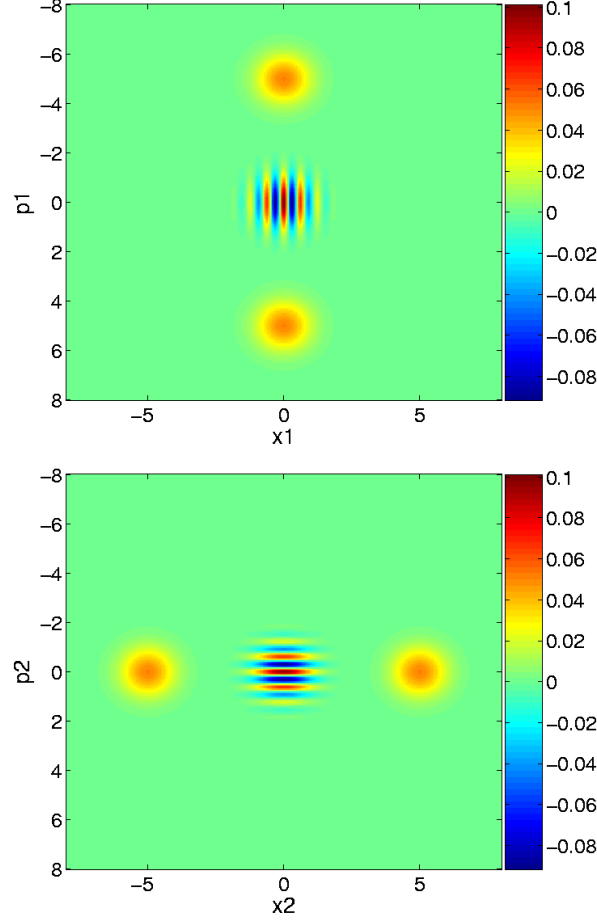


FIG. 2: (color online) Cross-sectional view of Wigner function with parameter value as in Fig.(1) except the entanglement has been turned off by setting the parameter $B = 0$.

we have calculated above. These plots look very similar to that in [1], but no oblique sidebands, as seen in [1], is visible in our figure. The oblique side bands in our case come from the off diagonal terms W_{C1} and W_{C2} . As seen in Eq.(9) they are multiplied by constant Gaussian factors and their contribution to the Wigner function is strongly suppressed for sufficiently large values of x_0 and p_0 . In this sense, we observe a cleaner checker-board type pattern using the bipartite compass state.

Fig.(2) depicts the case when $B = 0$ and all the other parameters are same as in Fig.(1). No sub-Planck structure is seen here, which confirms our assertion that both A and B must be simultaneously non-zero to have these phase-space structures in x_1p_1 and x_2p_2 planes. We have chosen $x_0, p_0=5$, in the units of $\hbar = 1$ and $\delta = 1$ in plotting Figs.(1-2). The circles indicate the positions of the Gaussians. For $B = 0$ case, only two Gaussians are visible.

Sensitivity of the entangled compass state in Eq.(3) can be studied as follows. Let $D_1(\alpha)$ and $D_2(\beta)$ denote two displacement operators causing the displacement of particle states one and two, by amount α and β respectively to create a perturbed state $|\psi_{per}\rangle = D_1(\alpha)D_2(\beta)|\psi_c\rangle$. The overlap function $|\langle\psi_c|\psi_{per}\rangle|^2$ can be found to be

$$|\langle\psi_c|\psi_{per}\rangle|^2 = 16|\mathcal{N}|^4 [|A|^4 \cos^2 \{x_0(\beta + \beta^*)\} \cosh^2 \{x_0(\alpha^* - \alpha)\} + |B|^4 \cos^2 \{x_0(\beta + \beta^*)\} \cosh^2 \{x_0(\alpha^* - \alpha)\} \\ + 2|A||B| \cos \{x_0(\beta + \beta^*)\} \cosh \{x_0(\alpha^* - \alpha)\} \cos \{x_0(\beta + \beta^*)\} \cosh \{x_0(\alpha^* - \alpha)\}]. \quad (14)$$

It is particularly of interest to consider the case when there is equal shift to both the particles i.e., $\alpha = \beta = is \frac{x_0}{|x_0|}$, the overlap function can then be written as

$$8|\mathcal{N}|^4 \{1 + \cos(4x_0s)\}.$$

Clearly the overlap function becomes minimum, for the distinguishable displacement, if $s \sim \pi/(4x_0)$. Next consider

the displacement of the compass state given in Ref.[1, 2], by amount s_1 , the overlap function can be written as

$$\frac{1}{4} \{3 + 4\cos(2x_0s_1) + \cos(4x_0s_1)\} \quad (15)$$

This function becomes minimum for $s_1 \sim \pi/(2x_0)$. Thus the distinguishable displacement coming from the entangled state is a factor 1/2 less than the one found in Ref.[2]. Since the sensitivity to the measurement depends upon $\frac{1}{x_0}$, both the states, the one given in Refs.[1, 2] and in Eq.(3), can be useful in carrying out the Heisenberg limited measurements. The energy resource required by the present state is less than that required in the nonentangled scenario.

In conclusion, we have studied the phase-space structures in a bipartite system of entangled superposed coherent-states. It was shown that the Wigner function for the quantum state have sub-Planck structures arising due to entanglement. The Wigner function, in the four dimensional phase-space, have these structures in x_1p_1 , x_1p_2 , x_2p_1 and x_2p_2 planes. But we have argued that patterns seen in the off-diagonal planes in the phase-space may not be physically relevant. They exist with or without entanglement. The structures seen in the diagonal planes are induced by entanglement and can be physically important. Entanglement makes them a better compass state in the sense that they are robust against decoherence. Furthermore, these structures are cleaner in this bipartite system due to the suppression of the side bands. We have shown that this kind of compass state may be useful in carrying out precession quantum measurements with less energy resource.

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